

ON THE WARING-GOLDBACH PROBLEM FOR TENTH POWERS

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1. INTRODUCTION

Define $H(k)$ to be the least s such that all sufficiently large n satisfying some congruence condition are the sum of s k^{th} powers of primes.

Work done by Vinogradov [17], Hua [3, 4], and Davenport [1], and later work by Thanigasalam [10, 11, 12] and Vaughan [15] resulted in the knowledge that

$$H(4) \leq 15, H(5) \leq 23, H(6) \leq 33, H(7) \leq 47, H(8) \leq 63, H(9) \leq 83, H(10) \leq 107.$$

Recently, Kawada and Wooley [5] and Kumchev [6] showed that

$$H(4) \leq 14, H(5) \leq 21, H(7) \leq 46,$$

and even more recently, Zhao [18] has shown that

$$H(4) \leq 13, H(6) \leq 32.$$

The purpose of this paper is to establish similar results for tenth powers.

In particular, we shall obtain the following:

Theorem 1.¹

$H(10) \leq 105$. *In particular, every sufficiently large integer congruent to 6 modulo 33 is the sum of 105 tenth powers of primes.*

The above result will be established with the Hardy-Littlewood method. The results which allow us to improve the previous bound of 107 are the new Weyl sum estimates obtained by Kumchev [7] and the mean value estimates in [13].

2. PRELIMINARIES

2.1. Notation. In all cases, unless otherwise specified, p will refer to a prime.

For $x \in \mathbb{R}$, let $e(x) = e^{2\pi i x}$.

As usual, we will write $O(f)$ to denote some quantity bounded above by $C|f|$ for some C , and $f \ll g$ means $|f| \leq Cg$ for some C , and ϵ will refer to a sufficiently small positive real number. All sums will be over the natural numbers unless otherwise specified.

We write

$$\mathfrak{M}(q, a; Q, P) = \left\{ \alpha \in [0, 1] : |q\alpha - a| \leq \frac{Q}{P} \right\}.$$

We define the primorial $Q\#$ of $Q > 0$ to be the product of all primes less than or equal to Q . Also, c will refer to some constant, and will not necessarily be the same each time it is mentioned in the paper.

Call a set $\{\lambda_1, \dots, \lambda_k\} \subset \mathbb{R}^+$ admissible if the number of solutions S to

$$\sum_{i \leq k} (x_i^k - y_i^k) = 0$$

satisfying $P^{\lambda_i} < x_i, y_i \leq 2P^{\lambda_i}$ satisfies $S \ll P^{\lambda_1 + \dots + \lambda_k + \epsilon}$.

¹Since the writing of this paper, the bound $H(10) \leq 89$ has been achieved in [9]

2.2. Exponential Sum Estimates.

Lemma 2. *For all α for which for all coprime $0 \leq a \leq q \leq P^{1/4}$ s.t. $|q\alpha - a| > P^{1/4}P^{-10}$,*

$$(2.1) \quad \sum_{P < p \leq 2P} e(\alpha p^{10}) \ll P^{1-1/480+\epsilon}$$

Proof. We have that by Theorem 2 in Kumchev [7],

$$\sum_{n \leq P} \Lambda(n) e(\alpha n^{10}) = \sum_{p \leq P} \log p e(\alpha p^{10}) + O(\sqrt{P}) \ll P^{1-1/480+\epsilon}.$$

Then, by partial summation, we have that

$$\sum_{p \leq P} e(\alpha p^{10}) = \frac{1}{\log P} \sum_{p \leq P} \log p e(\alpha p^{10}) - \int_2^P \left(-\frac{1}{t \log^2 t} \right) \sum_{p \leq t} \log p e(\alpha p^{10}) dt.$$

Note that

$$\begin{aligned} & \int_2^P \left(\frac{1}{t \log^2 t} \right) \sum_{p \leq t} \log p e(\alpha p^{10}) dt \leq \\ & \left| \int_2^P \left(\frac{1}{t \log^2 t} \right) \sum_{p \leq t} \log p e(\alpha p^{10}) dt \right| \ll \int_2^P \frac{t^\epsilon dt}{t^{1/480} \log^2 t} \ll P^{1-1/480+\epsilon}. \end{aligned}$$

The desired result follows. □

Lemma 3. *For some a, q, α satisfying $(a, q) = 1$, $|q\alpha - a| \leq QP^{-10}$ for some $q \leq Q \leq P$,*

$$\sum_{P < p \leq 2P} e(\alpha p^{10}) \ll q^\epsilon (\log P)^c \left(P (q + P^{10}|q\alpha - a|)^{-1/2} + P^{11/20} (q + P^5|q\alpha - a|)^{-1/2} \right).$$

Proof. This is just Lemma 5.6 in Kumchev [6] with $M = 1/2$, $z = \sqrt{2P}$. □

3. MEAN-VALUE ESTIMATES

Lemma 4. *There exist admissible exponents $1 = \lambda_1, \dots, \lambda_{51}$ satisfying*

$$(3.1) \quad \alpha_{51} = \frac{\lambda_1 + \dots + \lambda_{51}}{10} > 0.999553 > 1 - \frac{1}{2230}.$$

Proof. This follows from Lemma 18 in Thanigasalam [14] and (10.5) in Thanigasalam [12]. □

For $1 \leq j \leq 51$, write $P_j = P^{\lambda_j}$ where the λ_i are as in Lemma 4, and let

$$f_j(\alpha) = \sum_{P_j < n \leq 2P_j} e(\alpha n^{10})$$

Then, since the λ_j are admissible, we have that the following, which is the main result of this section holds:

Lemma 5. *We have that*

$$(3.2) \quad \int_0^1 |f_1(\alpha) \dots f_{51}(\alpha)|^2 d\alpha \ll P^{10\alpha_{51}+\epsilon},$$

where α_{51} is the constant mentioned in Lemma 4.

4. PROOF OF THE MAIN THEOREM

Let N be some large integer congruent to 6 (mod 33), let B be a sufficiently large real number, and set

$$P = \frac{1}{2}N^{1/10}, \quad X = P_1^5 P_2^2 \dots P_{51}^2 N^{-1}, \quad L = \log^B P.$$

Let

$$\mathfrak{N}(q, a) = \mathfrak{M}(q, a; L, P^{10}) \quad \mathfrak{N} = \bigcup_{\substack{0 \leq a \leq q \leq \log^B P \\ (a, q) = 1}} \mathfrak{N}(q, a)$$

$$\mathfrak{M} = \bigcup_{\substack{0 \leq a \leq q \leq P^{1/4} \\ (a, q) = 1}} \mathfrak{M}(q, a; P^{1/4}, P^{10})$$

and let $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$, $\mathfrak{n} = [0, 1) \setminus \mathfrak{N}$

For $1 \leq i \leq 51$, define

$$g_i(\alpha) = \sum_{P < p \leq 2P} e(\alpha p^{10}).$$

For some measurable $\mathfrak{B} \subseteq [0, 1)$, write

$$R(N; \mathfrak{B}) = \int_0^1 g_1(\alpha)^5 g_2(\alpha)^2 \dots g_{51}(\alpha)^2 d\alpha.$$

Let

$$R(N) = |\{(p_1, \dots, p_{105}) : p_1^{10} + \dots + p_{105}^{10} = N\}|,$$

for primes p_1, \dots, p_{105} satisfying

$$P_1 < p_1, p_2, p_3, p_4, p_5 \leq 2P_1, P_i < p_{2i+2}, p_{2i+3} \leq 2P_i \text{ for } 2 \leq i \leq 51.$$

Then, by orthogonality, we have that $R(N) = R(N; [0, 1))$.

Note that in order to prove Theorem 1, it is sufficient to show that for all sufficiently large N , $R(N) > 0$.

4.1. The major arcs. In this section, we shall consider the contribution to $R(N)$ from the major arcs \mathfrak{N} . Write

$$S(q, a) = \sum_{\substack{1 \leq k \leq q \\ (k, q) = 1}} e\left(\frac{ak^{10}}{q}\right),$$

$$v_i(\beta) = \int_{P_i}^{2P_i} \frac{e(\beta t^{10})}{\log t} dt,$$

$$B(N, q) = \frac{1}{\phi(q)^{105}} \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} S(q, a) e\left(\frac{-aN}{q}\right),$$

$$\mathfrak{S}(N) = \sum_q B(n, q),$$

$$J(N; \xi) = \int_{-\xi}^{\xi} v_1(\beta)^5 v_2(\beta)^2 \dots v_{51}(\beta)^2 e(-N\alpha) d\beta,$$

and let $J(N) = J(N; \infty)$.

Note that by Theorem 12 in [4], $\mathfrak{S}(N) \asymp 1$. We also have that $J(N) \asymp X \log^{-105} P$.

We then have that by partial summation and the Siegel-Walfisz Theorem that for $\alpha \in \mathfrak{N}_0(q, a)$

$$g_i(\alpha) = \phi(q)^{-1} S(q, a) v(\alpha - a/q) + O(P_i L^{-3})$$

Therefore, since the measure of \mathfrak{N} is $O(L^2 n^{-1})$

$$R(N; \mathfrak{N}) = \mathfrak{S}(N) J(N) + O(X L^{-1}) \gg X \log^{-105} P$$

4.2. **The minor arcs.** In this section we shall bound the contribution from $\mathfrak{K} = \mathfrak{M} \cap \mathfrak{n}$ and \mathfrak{m}

Lemma 6. *There exists $\eta > 0$ s.t.*

$$(4.1) \quad R(N; \mathfrak{m}) \ll XP^{-\eta+\epsilon}$$

.

Proof. In fact, we shall prove that this is the case for all $\eta = 1/160 - 1/223$. We have that by (2.1),

$$\sup_{\alpha \in \mathfrak{m}} |g_1(\alpha)| \ll P^{1-1/480+\epsilon}.$$

It then follows from (3.1) and (3.2) by considering the underlying diophantine equation that

$$\begin{aligned} R(N; \mathfrak{m}) &= \int_{\mathfrak{m}} g_1(\alpha)^5 g_2(\alpha)^2 \dots g_{51}(\alpha)^2 d\alpha \\ &\ll \left(\sup_{\alpha \in \mathfrak{m}} |g_1(\alpha)| \right)^3 \int_0^1 |f_1(\alpha) \dots f_{51}(\alpha)|^2 d\alpha \ll P^{3-1/160} (P_1 \dots P_{51})^2 P^{10\alpha_{51}+\epsilon} \\ &\ll P^{1/223-1/160+\epsilon} P^{-10} (P_1 \dots P_{51})^2 \ll XP^{-\eta+\epsilon} \end{aligned}$$

as desired. □

Lemma 7. *We have that*

$$R(N; \mathfrak{K}) \ll XL^{-1} \log^c P.$$

Proof. Note that \mathfrak{K} is the disjoint union of $\mathfrak{K}(q, a)$ for coprime a, q satisfying $0 \leq a \leq q \leq P^{1/4}$, where $\mathfrak{K}(q, a) = \mathfrak{M}(q, a) \setminus \mathfrak{N}(q, a)$ for $q \leq L$ and $\mathfrak{K}(q, a) = \mathfrak{M}(q, a)$ otherwise. Then, it follows by applying Lemma 4 that

$$\begin{aligned} &\int_{\mathfrak{K}} g_1(\alpha)^5 g_2(\alpha)^2 \dots g_{51}(\alpha)^2 d\alpha \\ &\ll \int_{\mathfrak{K}} |g_1(\alpha)|^5 |g_2(\alpha)|^2 |g_3(\alpha)| |g_4(\alpha)| \dots |g_{51}(\alpha)|^2 d\alpha \\ &\ll Xn \sum_{q \leq P^{1/4}} \sum_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \int_{\mathfrak{K}(q, a)} \frac{\log^c P d\alpha}{q^4 (1 + n|\alpha - a/q|)^2}. \end{aligned}$$

The desired result follows. □

Now, it follows from this and (4.1), by making B sufficiently large, that

$$R(N) = R(N; \mathfrak{N}) + R(N; \mathfrak{K}) + R(N; \mathfrak{m}) \gg X \log^{-105} P,$$

so Theorem 1 holds.

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REFERENCES

- [1] H. Davenport, *On Waring's problem for fourth powers*, Ann. of Math **40** (1939).
- [2] N. J. de Bruijn *On the number of uncanceled elements in the sieve of Eratosthenes*, Proc. Kon. Ned. Akad. Wetensch, **53** (1950)
- [3] L. K. Hua, *Some results in prime number theory*, Quart. J. Math. Oxford Ser. **9** (1938).
- [4] L. K. Hua, *Additive Theory of Prime Numbers*, American Mathematical Society, Providence, RI, 1965.
- [5] K. Kawada, T.D. Wooley, *On the Waring-Goldbach problem for fourth and fifth powers*, Proc. London Math Soc. (3) **83** (2001)
- [6] A. Kumchev, *On the Waring-Goldbach problem for seventh powers*, Proc. Amer. Math. Soc. **133** (2005)
- [7] A. Kumchev, *On Weyl Sums over primes in short intervals*, Number theory arithmetic in ShangriLa, Ser. Number Theory Appl., vol. 8, World Sci. Publ., Hackensack, NJ, 2013, pp. 116131
- [8] A. Kumchev, *On Weyl Sums over primes and almost primes*, Michigan Math. J. **54** (2006)
- [9] A. Kumchev, T. D. Wooley, *On the Waring-Goldbach problem for eighth and higher powers*, in preparation.
- [10] K. Thanigasalam, *Improvement on Davenport's iterative method and new results in additive number theory, I*, Acta Arith. **46** (1985)
- [11] K. Thanigasalam, *Improvement on Davenport's iterative method and new results in additive number theory, II. Proof that $G(5) \leq 22$* , Acta Arith. **46** (1986)
- [12] K. Thanigasalam, *Improvement on Davenport's iterative method and new results in additive number theory, III*, Acta Arith. **48** (1987)
- [13] K. Thanigasalam, *On admissible exponents for k th powers*, Bull. Calcutta Math. Soc. **86** (1994)
- [14] K. Thanigasalam, *On Waring's Problem*, (1980)
- [15] R. C. Vaughan, *On Waring's problem for smaller exponents*, Proc. London Math. Soc. (3) **52** (1986).
- [16] R.C. Vaughan, *The Hardy-Littlewood Method*, second edition, Cambridge University Press, Cambridge, 1997.
- [17] I. M. Vinogradov, *Representation of an odd number as a sum of three primes*, C. R. Acad. Sci. URSS **15** (1937).
- [18] L. Zhao, *On the Waring-Goldbach problem for fourth and sixth powers*, Proc. Lon. Math. Soc. (6) **108** (2014)